Phase determination method in statistical generalized phase-shifting digital holography

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A simple estimation method of the relative phase shift for generalized phase-shifting digital holography based on a statistical method is proposed. This method consists of a selection procedure of an optimum cost function and a simple root-finding procedure. The value and sign of the relative phase shift are determined using the coefficient and the solution of the optimum cost function. The complex field of an object wave is obtained using the estimated relative phase shift. The proposed method lifts the typical restriction on the range of the phase shift due to the phase ambiguity problem. Computer simulations and optical experiments are performed to verify the proposed method.

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1. Introduction

Digital holography is a technique in which the interference pattern between a light beam diffracted from an object and a reference beam is recorded by an image sensor, such as a charge coupled device (CCD), and then processed digitally to reconstruct the entire object wavefront using, e.g., a Fresnel transform [1,2]. Phase-shifting digital holography has been developed to avoid the problem of overlap between the object wavefront and other components and to effectively exploit the limited spatial resolution of digital cameras [3]. The standard phase-shifting algorithm requires three or more digital holograms recorded using a phase-shifted reference wave. The step widths are typically multiples of $\pi/2$. However, the actual phase-shift value is typically slightly different to the theoretical value because of phase-shift errors due to, for example, the nonlinear properties of the phase shifter and adjustment errors. Therefore, strict phase-shifter calibration techniques should be introduced to suppress the phase-shift errors [4].

Generalized phase-shifting techniques that do not require a strict phase calibration have also been developed. In particular, a statistical generalized phase-shifting technique may be a simple and effective method for extracting the phase-shift value because the procedure and the optical system are almost the same as in the standard three-step phase-shifting method and the calculation for phase estimation is simple [5–11]. However, the statistical method usually assumes that the statistical property of the diffraction field of the object is fully random and that the phase shift ranges from 0 to π to avoid phase ambiguity of the inverse trigonometric function. The first condition may be ensured if the target object has a complex phase distribution because sufficient developed random phase distribution is often observed in the Fresnel diffraction field in the holography experiment. On the other hand, the second condition may be in conflict with the advantage such that the generalized phase-shifting technique does not require strict calibration and control of the phase shifter. This restriction of phase shift may limit the use of the statistical method in some applications, such as microscopy [12] and shape measurement [13].

In this paper, we propose a simple method for estimating the value and sign of the relative phase shift for generalized phase-shifting digital holography based on the statistical method. In the proposed

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method, a statistical approach is used to extract the relative phase-shift value without determining its sign. Then, the correct sign of the relative phase-shift value is determined by a simple root-finding method using the optimum cost function that is selected by the proposed algorithm. The fully complex field of an object wave is obtained using the relative phase-shift value with the correct sign. In Section 2, the principle of the phase estimation method for the statistical generalized phase-shifting digital holography is described. We confirm our algorithm by numerical simulation in Section 3. Experimental results are presented using a phase-shifting digital holography system with a phase shifter that has a coarse resolution in Section 4 and conclusions are given in Section 5.

2. Principle of the Method

Let us consider an optical configuration for in-line digital holography as shown in Fig. 1. Here, we assume that an object is placed at a distance d from the hologram plane and that $A(x,y) = |A(x,y)| \exp\{j\theta(x,y)\}$ and $R_i(x,y) = |R| \exp(j\phi_i)$ are the complex amplitudes of the object and reference plane-wave on the hologram plane, respectively, where the amplitude of $R_i(x,y)$ is constant and ϕ_i is the ith constant phase shift. We can represent three phase-shifted holograms as follows:

$$I_i(x,y) = |A|^2 + |R|^2 + 2|A||R|\cos(\theta - \phi_i), \quad (i = 0,1,2), \tag{1}$$

where the coordinate variable is omitted for simplicity.

Next, we calculate the difference between the pth and qth holograms as follows:

$$\begin{aligned}\Delta I_{pq} &= I_q(x,y) - I_p(x,y) \\ &= 4|A||R|\sin\frac{\Delta\phi_{pq}}{2}\sin\left(\theta - \frac{\phi_p + \phi_q}{2}\right), \\ &(p,q = 0,1,2),\end{aligned} \tag{2}$$

where $\Delta\phi_{pq} = \phi_q - \phi_p$. It should be noted that ΔI_{pq} consists of a constant coefficient that includes the relative phase shift and a sinusoidal function that represents the interference term. We consider the average of the absolute square value of the

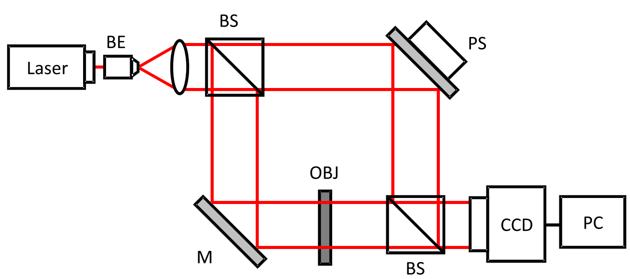

Fig. 1. (Color online) Optical configuration: BE, beam expander; BS, beam splitter; M, mirror; PS, phase shifter; and OBJ, object.

subtraction hologram for the whole hologram, which can be written as follows:

$$\begin{aligned}E_{pq} &= \langle|\Delta I_{pq}|^2\rangle \\ &= 8|R|^2\sin^2\frac{\Delta\phi_{pq}}{2}\{\langle|A|^2\rangle \\ &\quad - \langle|A|^2\sin(2\theta - \phi_p - \phi_q)\rangle\},\end{aligned} \tag{3}$$

where $\langle\rangle$ is the averaging operator over the whole frame. If the distance from the object plane to the hologram plane is sufficiently large, the phase of the object wave on the hologram plane can be considered a spatially random distribution owing to the Fresnel diffraction. Supposing that the statistical properties of the diffraction field correspond to those of a fully random field, the second term in Eq. (3) reduces to zero and we can write the following expression:

$$E_{pq} = 8|R|^2\sin^2\frac{\Delta\phi_{pq}}{2}\langle|A|^2\rangle. \tag{4}$$

Now, we introduce a positive parameter κ, which we define as $\kappa \equiv (4|R|^2\langle|A|^2\rangle)^{-1}$. Consequently, the relative phase-shift value can be derived as

$$\Delta\phi_{pq} = \arccos(1 - \kappa E_{pq}). \tag{5}$$

The relative phase-shift value can be obtained as a principal value within a range $[0, \pi]$ due to the properties of the arccosine function. Therefore, the phase shift must be restricted to within this range in order to obtain the correct phase-shift value. To avoid this restriction, the sign of the relative phase shift must be determined appropriately.

In order to determine the sign, we consider a new constraint condition. Because measurements of the phase are only available modulo 2π, not only the phase but also the relative phase-shift value generally has 2π ambiguity. Therefore, the total summation of the cyclic relative phase shift can be described as

$$\Delta\phi_{01} + \Delta\phi_{12} + \Delta\phi_{20} = 2m\pi, \tag{6}$$

where m is an integer. In this study, we refer to this constraint condition as a cyclic phase constraint condition. Now, we consider the cost function as

$$f_n(\kappa) = c_{01}\Delta\phi_{01} + c_{12}\Delta\phi_{12} + c_{20}\Delta\phi_{20}, \qquad c_{pq} = \pm 1, \tag{7}$$

where the coefficient corresponds to the sign of the phase shift since $\Delta\phi_{pq}$ is positive because of the nature of the arccosine function. If the cost function equals $2m\pi$, then the cost function satisfies the cyclic phase constraint condition and its coefficient is the correct sign of the phase shift. There are eight combinations of signs.

$$f_1 = \Delta\phi_{01} + \Delta\phi_{12} + \Delta\phi_{20},$$
$$f_2 = \Delta\phi_{01} + \Delta\phi_{12} - \Delta\phi_{20},$$
$$f_3 = \Delta\phi_{01} - \Delta\phi_{12} + \Delta\phi_{20},$$
$$f_4 = \Delta\phi_{01} - \Delta\phi_{12} - \Delta\phi_{20},$$
$$f_5 = -\Delta\phi_{01} - \Delta\phi_{12} - \Delta\phi_{20} (= -f_1),$$
$$f_6 = -\Delta\phi_{01} - \Delta\phi_{12} + \Delta\phi_{20} (= -f_2),$$
$$f_7 = -\Delta\phi_{01} + \Delta\phi_{12} - \Delta\phi_{20} (= -f_3),$$
$$f_8 = -\Delta\phi_{01} + \Delta\phi_{12} + \Delta\phi_{20} (= -f_4). \quad (8)$$

However, half of the cost functions are equivalent to sign inversions of the other half of the cost functions. This means that each pair of corresponding cost functions has opposing rotational directions of the phase. Therefore, we need only confirm four types of cost function.

We confirmed the value of the four cost functions within the range of κ by computer simulation as shown in Fig. 2. The range of κ can be deduced according to the condition enforced by the nature of the arccosine function, which is $-1 \leq 1 - \kappa E_{pq} \leq 1$, that is, $0 < \kappa \leq 2/E_{pq}$. If the maximum of E_{pq} is defined as $E_{\max} = \max[E_{01}, E_{12}, E_{20}]$, the range of κ can be obtained as

$$0 < \kappa \leq \frac{2}{E_{\max}}. \quad (9)$$

Figure 2(a) shows the cost function as a function of κ when the correct combination of signs is assumed to be f_1. Note that only the cost function $f_1(\kappa)$ crosses the horizontal axis $f_n(\kappa) = 0$, where $f_1(\kappa) - 2\pi$ is also plotted to illustrate the principal value. Similarly, if the correct combination of signs is f_n for $n = 2$, 3, and 4, the cost function $f_n(\kappa)$ crosses the horizontal axis $f_n(\kappa) = 0$. Figure 2(b) shows the result for $n = 3$. Consequently, only one of the four cost functions completely satisfies the cyclic phase constraint condition within the range of κ. Therefore, the optimum cost function can be selected by evaluating the zero-crossing property of the cost function.

The selection procedure for the optimum cost function is performed as follows. Supposing that κ_0 is the solution of $f_n(\kappa) = 2m\pi$, and that α is near zero and satisfies $0 < \alpha < \kappa_0 < 2/E_{\max}$. We calculate the values of the cost function at $\kappa = \alpha$ and $\kappa = 2/E_{\max}$. In order to consider a principal value, the cost function can be written as $f(\kappa) = f_1(\kappa) - 2\pi$ for $n = 1$, and $f(\kappa) = f_n(\kappa)$ for $n = 2$, 3, and 4. Then, the optimum cost function can be determined by evaluating the zero-crossing condition, which is that the sign of the cost function changes around κ_0, i.e., $f(\alpha)f(2/E_{\max}) < 0$.

Using the optimum cost function, the solution κ_0 can be obtained by finding a zero-crossing point by means of a simple root-finding method, such as the bisection method or the secant method. Consequently, the value and sign of the relative phase shift can be obtained by

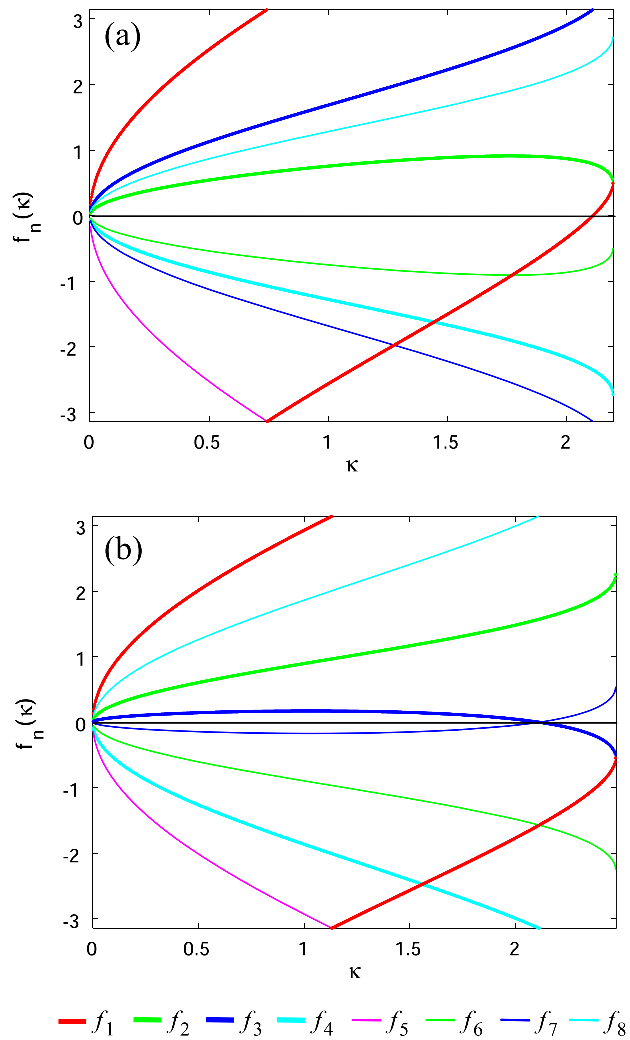

Fig. 2. (Color online) Variation of cost functions within the available range of κ when the optimum cost function is (a) $f_1(\kappa)$ and (b) $f_3(\kappa)$, where $f_1(\kappa) - 2\pi$ is also illustrated because the principal value is considered.

$$\Delta\phi_{pq} = \hat{c}_{pq} \arccos(1 - \kappa_0 E_{pq}), \quad (10)$$

where $\hat{c}_{pq}$ is the coefficient of the optimum cost function.

The object wave in the hologram plane can be obtained after some algebraic manipulation of Eq. (1) as follows:

$$A(x,y) = \frac{e^{j\phi_0}}{|R|\Delta}\{(1 - e^{-j\Delta\phi_{20}})\Delta I_{01} + (1 - e^{j\Delta\phi_{01}})\Delta I_{20}\}, \quad (11)$$

where $\Delta = -2j(\sin\Delta\phi_{01} + \sin\Delta\phi_{12} + \sin\Delta\phi_{20})$, and ϕ_0 is an initial phase that can be omitted without any loss of generality. The complex field of the object wave in the object plane can be calculated by the inverse Fresnel transform, by distance d, of the object wave obtained using the correct relative phase-shift value in the hologram plane. It is noted that the proposed algorithm cannot determine the rotational direction of the phase. If the opposite rotation of phase is used, the complex conjugate of the object wave is calculated by Eq. (11). However, it is not a serious problem because the object wave can be easily obtained using the sign inversion of the relative phase-shift value.

3. Numerical Simulation

To confirm the validity of the proposed method, we conduct a numerical simulation. The wavelength of the laser for illuminating the object is assumed to be 632.8 nm. The size of the image sensor is assumed to be 1024 × 1024 pixels with a 6.45 μm pixel pitch. For an object, we assume a spherical phase object placed 20 mm away from the image sensor as shown in Fig. 3(a). Three phase-shifted holograms are simulated using the reference plane-wave with preset arbitrary phase shifts. The diffracted field is calculated using the angular spectrum method [14].

We present a typical result of the proposed method. The absolute phase-shift values are assumed to be 0.0, 1.178, and 5.498 rad. Therefore, the relative phase-shift values are 1.178, 4.320, and −5.498 rad, and those principal values reduce to +1.178, −1.963, and +0.785, respectively. Figure 3(b) is one of the three phase-shifted digital holograms. A bright interference fringe is observed in the center area because diffracted light is collected by the effect of the spherical phase. Using the selection method with $\alpha = (2/E_{\max})/100$, the optimum cost function was selected to be $f_3(\kappa)$ and then $(c_{01}, c_{12}, c_{20}) = (+1, -1, +1)$. The solution κ_0 was obtained using the bisection method after about 15 iterations as shown in Fig. 4. Consequently, the relative phase-shift values were estimated to be +1.187, −1.977, and +0.790 rad, and the absolute estimation errors were 0.009, 0.014, and 0.005, respectively. The object wave in the hologram plane was obtained using Eq. (11). Figure 3(c) is a phase map reconstructed from the hologram obtained by using the proposed phase-shifting method. For comparison, the reconstructed phase map by the Fresnel transform alone is also shown in Fig. 3(d), where the hologram without the direct current term is used to confirm the difference of the reconstructed phase map and therefore the phase distortion is observed in the center area because of the interference between +1 and −1 order components. The results show that the proposed method can estimate the correct value and sign of the relative phase shift and that the object wave is clearly reconstructed without the zeroth-order and the conjugate wave. The proposed method enabled clearly reconstruction of the object wave

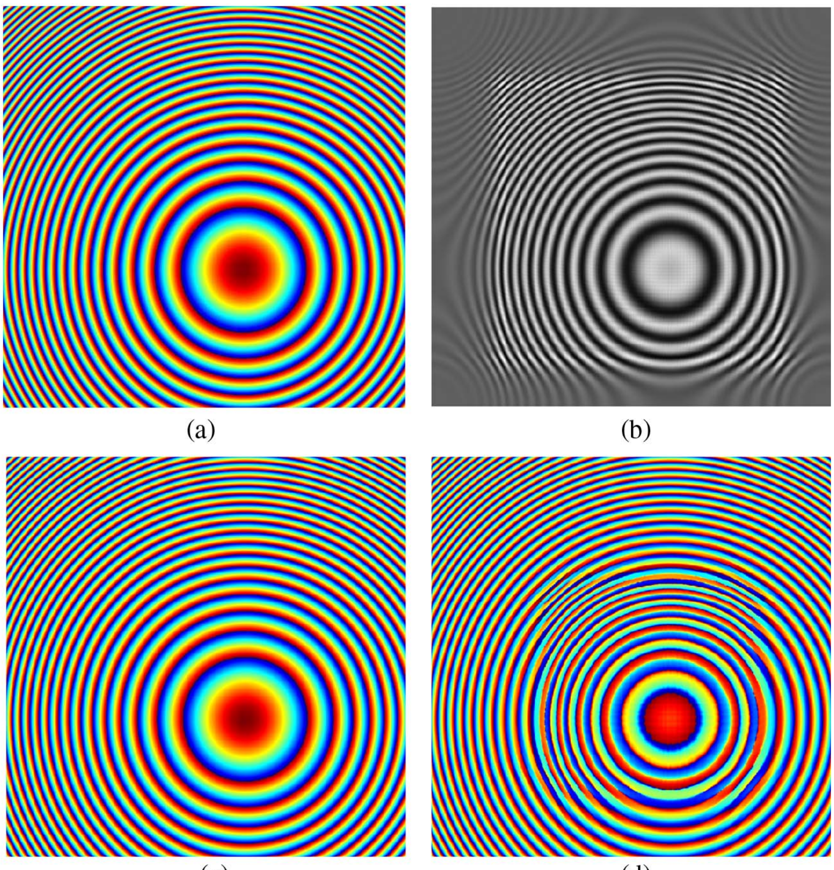

Fig. 3. (Color online) Computer simulations: (a) phase object, (b) typical phase-shifted digital hologram, (c) and (d) phase map reconstructed from the hologram obtained by using the proposed phase-shifting method and the Fresnel transform alone, respectively.

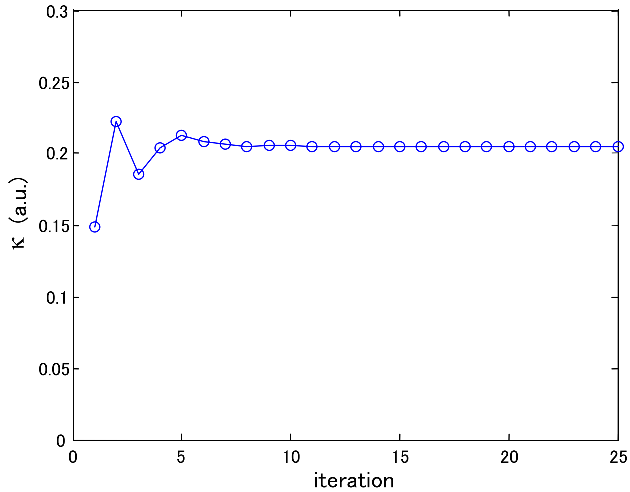

Fig. 4. (Color online) Determination of κ_0 using the bisection method.

even if the relative phase-shift value was assumed to be other arbitrary values including the multiples of $\pi/2$ that are used in the standard phase-shifting algorithm. However, when the relative phase-shift value was too small, that is, $\Delta\phi_{pq} \approx 0$, the proposed method did not work well because the modulation depth of the subtraction hologram shown in Eq. (2) became small.

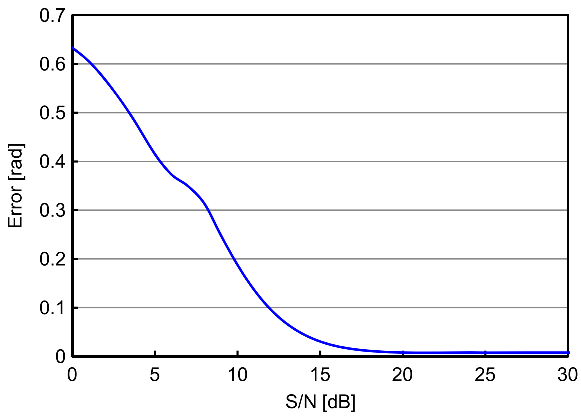

Fig. 5. (Color online) Phase estimation error as a function of the signal-to-noise ratio.

Next, we confirm the phase estimation error when the hologram is assumed to be corrupted by the addition of white Gaussian noise. The phase estimation error is defined as error $= 1/3 \sum(|\Delta\phi_{01} - \Delta\hat{\phi}_{01}| + |\Delta\phi_{12} - \Delta\hat{\phi}_{12}| + |\Delta\phi_{20} - \Delta\hat{\phi}_{20}|)$, where $\Delta\phi_{pq}$ and $\Delta\hat{\phi}_{pq}$ are original and estimated phase-shift values. Figure 5 shows the phase estimation error as a function of the signal to noise ratio. It can be seen that the phase estimation error can become small enough if the measurement can be performed with a sufficiently high signal to noise ratio. In this case, in order to obtain the phase estimation error under 0.01 rad, it is necessary to maintain the signal-to-noise ratio of over 20 dB.

4. Experimental Results

In this section, optical experiments are performed to verify the proposed method. We used a Mach–Zehnder-type interferometer as shown in Fig. 1. A He–Ne laser of 632.8 nm was used as a light source. The reference beam was a normal plane-wave. The phase shift of the reference wave was obtained by controlling a mirror mounted on a translation stage that has a minimum step of about 2 μm. The object was a green algae *volvox* placed at a distance of $d = 20$ mm away from the CCD camera. Three digital holograms with an arbitrary phase shift were recorded with 1024×1024 pixels by CCD camera with a square pixel 6.45 μm in size. Typical holograms are shown in Fig. 6.

Using the proposed method, the optimum cost function was selected to be $f_2(\kappa)$ and then $(c_{01}, c_{12}, c_{20}) = (+1, +1, -1)$. The relative phase-shift values were estimated to be +2.056, +0.743, and −2.799 rad by using the solution and coefficient of the optimum cost function. The object wave in the hologram plane was obtained using Eq. (11). Figures 7(a) and 7(b) show images of the numerically reconstructed intensity and phase. The magnified reconstructed images of the same area in the reconstructed images in Figs. 7(a) and 7(b) are shown in Figs. 7(c) and 7(d), respectively. The object wave was clearly reconstructed without the unwanted components. We can see that some spherical colonies in a hollow sphere can be observed. Note that the phase shifter cannot be precisely calibrated and controlled

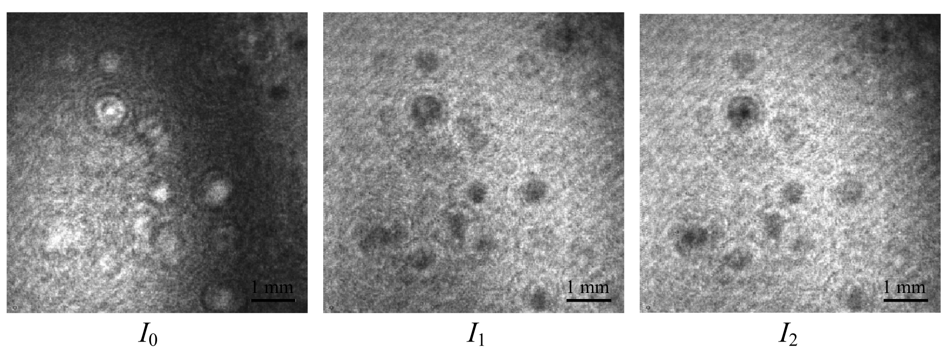

Fig. 6. Typical phase-shifted digital holograms.

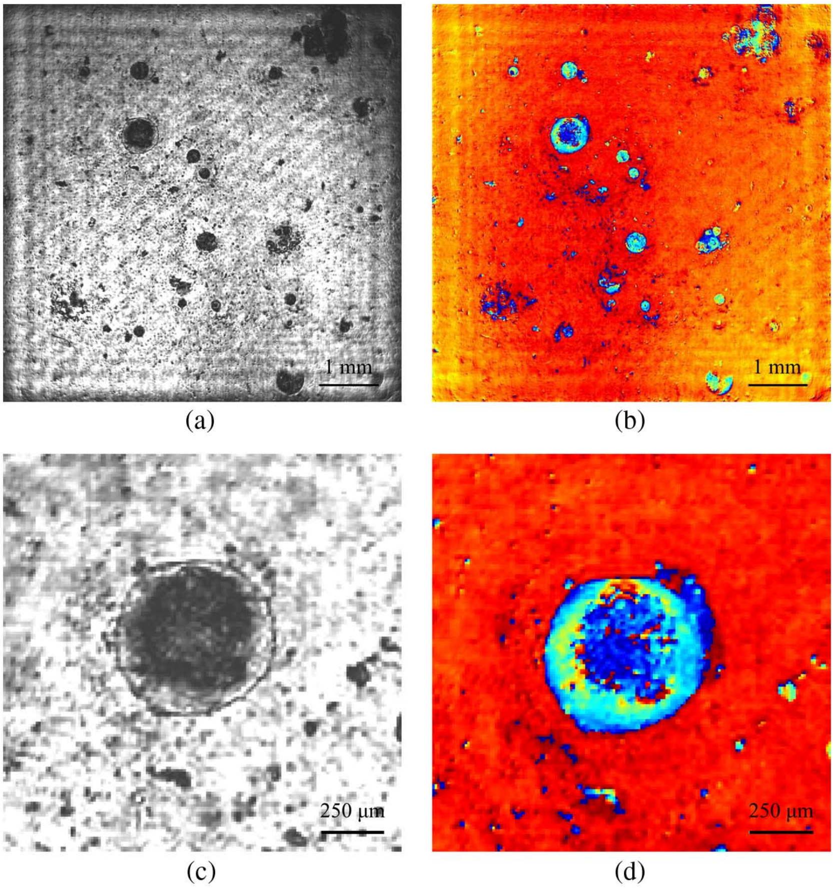

Fig. 7. (Color online) Numerically reconstructed (a) intensity and (b) phase; and (c) and (d) magnified reconstructed images of (a) and (b), respectively.

due to the coarse step size of the phase shifter in the experiment. These results indicate that the proposed method can be implemented using a simple and low-cost phase-shifting device.

5. Conclusions

We have proposed a new phase estimation method for statistical generalized phase-shifting digital holography. The value and sign of the relative phase shift can be estimated by imposing a specific constraint condition of the total summation of the cyclic relative phase shift on the phase-shift value obtained by the statistical method. The method consists of a selection procedure of an optimum cost function and a simple root-finding procedure. The object wave is obtained by using the correct phase-shift value calculated using the coefficient and solution of the optimum cost function. The method lifts the typical restriction imposed on the range of phase-shift values by the phase ambiguity problem. The method is easily implemented using a simple phase shifter because precise calibration and control of the phase shifter are not necessary.

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References

1. J. W. Goodman and R. W. Lawrence, "Digital image formation from electronically detected holograms," Appl. Phys. Lett. **11**, 77–79 (1967).
2. U. Schnars and W. Jüptner, "Direct recording of holograms by a CCD target and numerical reconstruction," Appl. Opt. **33**, 179–181 (1994).
3. I. Yamaguchi and T. Zhang, "Phase-shifting digital holography," Opt. Lett. **22**, 1268–1270 (1997).
4. H. Schreiber and J. H. Bruning, "Phase shifting interferometry," in *Optical Shop Testing*, D. Malacara, ed. (Wiley, 2007), pp. 547–666.
5. L. Z. Cai, Q. Liu, and X. L. Yang, "Phase-shift extraction and wave-front reconstruction in phase-shifting interferometry with arbitrary phase steps," Opt. Lett. **28**, 1808–1810 (2003).
6. L. Z. Cai, Q. Liu, and X. L. Yang, "Generalized phase-shifting interferometry with arbitrary unknown phase steps for diffraction objects," Opt. Lett. **29**, 183–185 (2004).
7. X. F. Xu, L. Z. Cai, X. F. Meng, G. Y. Dong, and X. X. Shen, "Fast blind extraction of arbitrary unknown phase shifts by an iterative tangent approach in generalized phase-shifting interferometry," Opt. Lett. **31**, 1966–1968 (2006).
8. X. F. Xu, L. Z. Cai, Y. R. Wang, X. L. Yang, X. F. Meng, G. Y. Dong, X. X. Shen, and H. Zhang, "Generalized phase-shifting interferometry with arbitrary unknown phase shifts: direct wave-front reconstruction by blind phase shift extraction and its experimental verification," Appl. Phys. Lett. **90**, 121124 (2007).
9. X. F. Xu, L. Z. Cai, Y. R. Wang, X. F. Meng, W. J. Sun, H. Zhang, X. C. Cheng, G. Y. Dong, and X. X. Shen, "Simple direct extraction of unknown phase shift and wavefront reconstruction in generalized phase-shifting interferometry: algorithm and experiments," Opt. Lett. **33**, 776–778 (2008).

10. X. F. Meng, L. Z. Cai, Y. R. Wang, X. L. Yang, X. F. Xu, G. Y. Dong, X. X. Shen, and X. C. Cheng, "Wavefront reconstruction by two-step generalized phase-shifting interferometry," Opt. Commun. **281**, 5701–5705 (2008).
11. P. Gao, B. Yao, N. Lindlein, K. Mantel, I. Harder, and E. Geist, "Phase-shift extraction for generalized phase-shifting interferometry," Opt. Lett. **34**, 3553–3555 (2009).
12. B. Kemper and G. V. Bally, "Digital holographic microscopy for live cell applications and technical inspection," Appl. Opt. **47**, A52–A61 (2008).
13. C. Wagner, S. Seebacher, W. Osten, and W. Jüptner, "Digital recording and numerical reconstruction of lensless Fourier holograms in optical metrology," Appl. Opt. **38**, 4812–4820 (1999).
14. J. W. Goodman, "*Introduction to Fourier Optics*," 2nd ed. (McGraw-Hill, 1996).